Homotopical Aspects of Commutative Algebras

Z. Arvasi and E. Ulualan

Abstract

This article investigates the homotopy theory of simplicial commutative algebras with a view to homological applications. [1]

Introduction

The original motivation for this article was to see what parts of the group theoretic case of crossed homotopical algebra generalised to the context of commutative algebras and to see how existing parts of commutative algebra might interact with the analogue. The hope was for a clarification of the group theoretic situation as well as perhaps introducing 'new' tools into commutative algebra.

Simplicial commutative algebras occupy a place somewhere between homological algebra, homotopy theory, algebraic K-theory and algebraic geometry. In each sector they have played a significant part in developments over quite a lengthy period of time. Their own internal structure has however been studied in [3, 4, 5]. The present work gives some lights on the 3-types of simplicial commutative algebras and will apply the results in various mainly homological settings.

Crossed modules of groups were introduced by Whitehead [25]. They model homotopy types with trivial homotopy groups in dimensions bigger than 2. Algebraic models for connected 3-types in terms of crossed module of groups of length 2 are 2-crossed module defined by Conduché [12] and crossed square defined by Guin-Walery, Loday [18]. The commutative algebra version of these structures are respectively defined by Grandjeán-Vale [17] and Ellis [14]. The first author and Porter (cf. [3, 4, 5]) also gave the relations between these constructions and simplicial algebras. For an alternative model, the notion of quadratic module of groups is defined by Baues (cf. [7]). This is a 2-crossed module with additional nilpotent conditions.

In this paper, we define the commutative algebra version of quadratic modules and construct a functor from the category of simplicial commutative algebras to that of quadratic modules by using higher dimensional Peiffer elements. In this construction we see the role of the hypercrossed complex pairings. Furthermore, we give a link between quadratic modules,

[1]Quadratic Modules, Simplicial Commutative Algebras, 2-Crossed Modules, Crossed Squares.

2-crossed modules, crossed squares and simplicial commutative algebras with Moore complex of length 2. Another algebraic model of 3-types is ‘braided regular crossed module’ introduced by Brown and Gilbert in [9]. This model will be analyzed in a separate paper.

Quillen [24] and Illusie [19] both discuss the basic homotopical algebra of simplicial algebras and their application in deformation theory. André [1] gives a detailed examination of their construction and applies them to cohomology via the cotangent complex construction.

Thus the situations which are examined in this paper and related models such as [3], [4] can be summarised in the following diagram

$$\begin{tikzcd}
\mathbf{QM} & & \mathbf{Crs}^2 \arrow[ll, "\Psi"'] \arrow[dl, leftrightarrow, "{[3]}"'] \arrow[ddl, leftrightarrow, "{[3]}"] \\
& \mathbf{SimpAlg}_{\leqslant 2} \arrow[ul, "\Delta"'] \arrow[d, leftrightarrow, "{[4]}"] & \\
& \mathbf{X}_2\mathbf{Mod} \arrow[uul, "\Lambda"] &
\end{tikzcd}$$

where the functors Λ, Δ, Ψ are given by Propositions 5.2, 6.1 and 7.1 respectively and the numbers in the diagram correspond to the references.

1 Preliminaries

In what follows ‘algebras’ will be commutative algebras over an unspecified commutative ring, $\mathbf{k}$, but for convenience are not required to have a multiplicative identity. The category of commutative algebras will be denoted by $\mathbf{Alg}$.

Simplicial Commutative Algebras

A simplicial (commutative) algebra $\mathbf{E}$ consists of a family of algebras $\{E_n\}$ together with face and degeneracy maps $d_i = d_i^n : E_n \to E_{n-1}, \quad 0 \leqslant i \leqslant n, \quad (n \neq 0)$ and $s_i = s_i^n : E_n \to E_{n+1}, \quad 0 \leqslant i \leqslant n$, satisfying the usual simplicial identities given in André [1] or Illusie [19] for example. It can be completely described as a functor $\mathbf{E} : \Delta^{op} \to \mathbf{Alg}$ where Δ is the category of finite ordinals $[n] = \{0 < 1 < \cdots < n\}$ and increasing maps. We denote the category of simplicial algebras by $\mathbf{SimpAlg}$.

Given a simplicial algebra $\mathbf{E}$, the Moore complex $(\mathbf{NE}, \partial)$ of $\mathbf{E}$ is the chain complex defined by

$$NE_n = \ker d_0^n \cap \ker d_1^n \cap \cdots \cap \ker d_{n-1}^n$$

with $\partial_n : NE_n \to NE_{n-1}$ induced from d_n^n by restriction.

The nth *homotopy module* $\pi_n(\mathbf{E})$ of $\mathbf{E}$ is the nth homology of the Moore complex of $\mathbf{E}$, i.e.

$$\pi_n(\mathbf{E}) \cong H_n(\mathbf{NE}, \partial) = \bigcap_{i=0}^{n} \ker d_i^n / d_{n+1}^{n+1} \left(\bigcap_{i=0}^{n} \ker d_i^{n+1} \right)$$

We say that the Moore complex **NE** of a simplicial algebra is of length k if $NE_n = 0$ for all $n \geqslant k+1$, so that a Moore complex of length k is also of length l for $l \geqslant k$. We denote thus the category of simplicial algebras with Moore complex of length k by $\mathbf{SimpAlg}_{\leqslant k}$.

The following terminologies and notations are derived from [10] and the published version, [11], of the analogous group theoretic case. For detailed investigation see [4].

For the ordered set $[n] = \{0 < 1 < \cdots < n\}$, let $\alpha_i^n : [n+1] \to [n]$ be the increasing surjective map given by

$$\alpha_i^n(j) = \begin{cases} j & \text{if } j \leqslant i, \\ j-1 & \text{if } j > i. \end{cases}$$

Let $S(n, n-r)$ be the set of all monotone increasing surjective maps from $[n]$ to $[n-r]$. This can be generated from the various α_i^n by composition. The composition of these generating maps is subject to the following rule: $\alpha_j \alpha_i = \alpha_{i-1}\alpha_j$, $j < i$. This implies that every element $\alpha \in S(n, n-r)$ has a unique expression as $\alpha = \alpha_{i_1} \circ \alpha_{i_2} \circ \cdots \circ \alpha_{i_r}$ with $0 \leqslant i_1 < i_2 < \cdots < i_r \leqslant n-1$, where the indices i_k are the elements of $[n]$ such that $\{i_1, \ldots, i_r\} = \{i : \alpha(i) = \alpha(i+1)\}$. We thus can identify $S(n, n-r)$ with the set $\{(i_r, \ldots, i_1) : 0 \leqslant i_1 < i_2 < \cdots < i_r \leqslant n-1\}$. In particular, the single element of $S(n,n)$, defined by the identity map on $[n]$, corresponds to the empty 0-tuple ( ) denoted by $\emptyset_n$. Similarly the only element of $S(n,0)$ is $(n-1, n-2, \ldots, 0)$. For all $n \geqslant 0$, let

$$S(n) = \bigcup_{0 \leqslant r \leqslant n} S(n, n-r).$$

We say that $\alpha = (i_r, \ldots, i_1) < \beta = (j_s, \ldots, j_1)$ in $S(n)$, if $i_1 = j_1, \ldots, i_k = j_k$ but $i_{k+1} > j_{k+1}, (k \geqslant 0)$ or if $i_1 = j_1, \ldots, i_r = j_r$ and $r < s$.
This makes $S(n)$ an ordered set.

Hypercrossed Complex Pairings

We give the following statements from [4]. For details see [11] (group case) and [4]. We define a set $P(n)$ consisting of pairs of elements (α, β) from $S(n)$ with $\alpha \cap \beta = \emptyset$ and $\beta < \alpha$ where $\alpha = (i_r, \ldots, i_1)$, $\beta = (j_s, \ldots, j_1) \in S(n)$. The **k**-linear morphisms that we will need,

$$\{C_{\alpha,\beta} : NE_{n-\#\alpha} \otimes NE_{n-\#\beta} \to NE_n : (\alpha, \beta) \in P(n), n \geqslant 0\}$$

are given as composites:

$$\begin{aligned} C_{\alpha,\beta}(x_\alpha \otimes y_\beta) &= p\mu(s_\alpha \otimes s_\beta)(x_\alpha \otimes y_\beta) \\ &= p(s_\alpha(x_\alpha) s_\beta(x_\beta)) \\ &= (1 - s_{n-1}d_{n-1}) \cdots (1 - s_0 d_0)(s_\alpha(x_\alpha) s_\beta(x_\beta)), \end{aligned}$$

where $s_\alpha = s_{i_r} \ldots s_{i_1} : NE_{n-\#\alpha} \to E_n$, $s_\beta = s_{j_s} \ldots s_{j_1} : NE_{n-\#\beta} \to E_n$, $p : E_n \to NE_n$ is defined by composite projections $p = p_{n-1} \ldots p_0$ with

$$p_j = 1 - s_j d_j \text{ for } j = 0, 1, \ldots, n-1$$

and $\mu : E_n \otimes E_n \to E_n$ denotes multiplication.

We will now consider that the ideal I_n in E_n such that generated by all elements of the form;

$$C_{\alpha,\beta}(x_\alpha \otimes y_\beta)$$

where $x_\alpha \in NE_{n-\#\alpha}$ and $y_\beta \in NE_{n-\#\beta}$ and for all $(\alpha, \beta) \in P(n)$.

Proposition 1.1 ([4]) *Let $\boldsymbol{E}$ be simplicial algebra and $n > 0$, and D_n the ideal in E_n generated by degenerate elements. We suppose $E_n = D_n$, and let I_n be the ideal generated by elements of the form*

$$C_{\alpha,\beta}(x_\alpha \otimes y_\beta) \qquad \text{with } (\alpha, \beta) \in P(n)$$

where $x_\alpha \in NE_{n-\#\alpha}, y_\beta \in NE_{n-\#\beta}$ with $1 \leqslant r, s \leqslant n$. Then,

$$\partial_n(NE_n) = \partial_n(I_n).$$

Now according to above proposition for $n = 2,\, 3$, we show the image of I_n by ∂_n what it looks like.

We suppose $D_2 = E_2$. We take $\beta = (1), \alpha = (0)$ and $x, y \in NE_1 = \ker d_0$. We know that the ideal I_2 is generated by elements of the form

$$C_{(1)(0)}(x \otimes y) = p_1p_0(s_1xs_0y) = s_1x(s_1y - s_0y).$$

Then the image of I_2 by ∂_2 is $d_2[C_{(1)(0)}(x\otimes y)] = x(y - s_0d_1y)$ where $x \in \ker d_0$ and $y - s_0d_1y \in \ker d_1$. Therefore the image of I_2 by ∂_2 is $\ker d_0 \ker d_1$.
For $n = 3$, the linear morphisms are

$$\begin{array}{lll} C_{(1,0),(2)}, & C_{(2,0),(1)}, & C_{(2,1),(0)} \\ C_{(2),(0)}, & C_{(2),(1)}, & C_{(1),(0)}. \end{array}$$

Then the ideal I_3 is generated by the elements; for $x \in NE_1$ and $y \in NE_2$

$$\begin{aligned} C_{(1,0),(2)}(x \otimes y) &= (s_1s_0x - s_2s_0x)s_2y \\ C_{(2,0),(1)}(x \otimes y) &= (s_2s_0x - s_2s_1x)(s_1y - s_2y) \\ C_{(2,0),(1)}(x \otimes y) &= s_2s_1x(s_0y - s_1y + s_2y) \end{aligned}$$

and for $x, y \in NE_2$

$$\begin{aligned} C_{(1),(0)}(x \otimes y) &= s_1x(s_0y - s_1y) + s_2xy \\ C_{(2),(0)}(x \otimes y) &= s_2xs_0y \\ C_{(1),(0)}(x \otimes y) &= s_2x(s_1y - s_2y). \end{aligned}$$

Thus the image of I_3 by ∂_3 is

$$
\begin{aligned}
\partial_3(C_{(1,0)(2)}(x\otimes y)) &= (s_1s_0d_1x - s_0x)y \in \ker d_2(\ker d_0\cap\ker d_1)\\
\partial_3(C_{(2,0)(1)}(x\otimes y)) &= (s_0x - s_1x)(s_1d_2y - y) \in \ker d_1(\ker d_0\cap\ker d_2)\\
\partial_3(C_{(2,1)(0)}(x\otimes y)) &= s_1x(s_0d_2y - s_1d_2y + y) \in \ker d_0(\ker d_1\cap\ker d_2)\\
\partial_3(C_{(2)(1)}(x\otimes y)) &= x(s_1d_2y - y) \in (\ker d_0\cap\ker d_1)(\ker d_0\cap\ker d_2)\\
\partial_3(C_{(2)(0)}(x\otimes y)) &= xs_0d_2y \in K_{\{0,1\}}K_{\{1,2\}} + K_{\{0,1\}}K_{\{0,2\}}\\
\partial_3(C_{(1)(0)}(x\otimes y)) &= s_1d_2xs_0d_2y - s_1d_2xs_1d_2y + xy\\
&\in K_{\{0,2\}}K_{\{1,2\}} + K_{\{0,1\}}K_{\{1,2\}} + K_{\{0,1\}}K_{\{0,2\}}.
\end{aligned}
$$

If $n=4$ then the image of the Moore complex of the simplicial algebra $\mathbf{E}$ can be given in the form

$$\partial_n(NE_n) = \sum_{I,J} K_IK_J,$$

where $\emptyset \neq I, J \subset [n-1] = \{0,1,\ldots,n-2\}$ with $I\cup J = [n-1]$, and where

$$K_I = \bigcap_{i\in I}\ker d_i \text{ and } K_J = \bigcap_{J\in J}\ker d_j.$$

In general for $n>4$, there is an inclusion

$$\sum_{I,J} K_IK_J \subset \partial_n(NE_n).$$

2 2-Crossed Modules from Simplicial Algebras

Crossed modules techniques give a very efficient way of handling information about on a homotopy type. They correspond to a 2-type. The commutative algebra analogue of crossed modules has been given by Porter in [23]. Throughout this paper we denote an action of $r\in R$ on $c\in C$ by $c\cdot r$.

A *crossed module* is an R-algebra homomorphism $\partial : C\to R$ with the action of R on C such that $\partial(c\cdot r) = \partial(c)r$ and $c'\cdot\partial(c) = c'c$ for all $r\in R$ and $c,c'\in C$. The second condition is called the *Peiffer identity*. We will denote such a crossed module by (C,R,∂). A morphism of crossed modules from (C,R,∂) to (C',R',∂') is a pair of $\mathbf{k}$-algebra homomorphisms, $\phi : C\to C'$ and $\varphi : R\to R'$ such that $\phi(c\cdot r) = \phi(c)\varphi(r)$. We thus define the category of crossed modules of commutative algebras denoting it by **XMod**.

As we mentioned in introduction, the notion 2-crossed modules of groups was introduced by Conduché in [12] as a model for connected 3-types. He showed that the category of 2-crossed modules is equivalent to the category of simplicial groups with Moore complex of length 2. Grandjeán and Vale, [17], gave the notion of 2-crossed modules on commutative algebras. Now, we recall from [17] the definition of a 2-crossed module:

A *2-crossed module* of $\mathbf{k}$-algebras consists of a complex of C_0-algebras

$$C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

and ∂_2, ∂_1 morphisms of C_0-algebras, where the algebra C_0 acts on itself by multiplication such that

$$\partial_2 : C_2 \longrightarrow C_1$$

is a crossed module. Thus C_1 acts on C_2 via C_0 and we require that for all $x \in C_2$, $y \in C_1$, $z \in C_0$ that $(xy)z = x(yz)$. Further there is a C_0-bilinear function giving

$$\{- \otimes -\} : C_1 \otimes_{C_0} C_1 \longrightarrow C_2$$

called a Peiffer lifting, which satisfies the following axioms

$$\begin{array}{lrcl}
\mathbf{2CM1}) & \partial_2\{y_0 \otimes y_1\} & = & y_0y_1 - y_0 \cdot \partial_1 y_1 \\
\mathbf{2CM2}) & \{\partial_2(x_1) \otimes \partial_2(x_2)\} & = & x_1x_2 \\
\mathbf{2CM3}) & \{y_0 \otimes y_1y_2\} & = & \{y_0y_1 \otimes y_2\} + \{y_0 \otimes y_1\} \cdot \partial_1 y_2 \\
\mathbf{2CM4}) & a)\{\partial_2 x \otimes y\} & = & x \cdot y - x \cdot \partial_1 y \\
 & b)\{y \otimes \partial_2 x\} & = & x \cdot y \\
\mathbf{2CM5}) & \{y_0 \otimes y_1\} \cdot z & = & \{y_0 \cdot z \otimes y_1\} = \{y_0 \otimes y_1 \cdot z\}
\end{array}$$

for all $x, x_1, x_2 \in C_2$, $y, y_0, y_1, y_2 \in C_1$ and $z \in C_0$.

A morphism of 2-crossed modules of algebras may be pictured by the diagram

$$\begin{tikzcd}
C_2 \arrow[r, "\partial_2"] \arrow[d, "f_2"'] & C_1 \arrow[r, "\partial_1"] \arrow[d, "f_1"'] & C_0 \arrow[d, "f_0"'] \\
C_2' \arrow[r, "\partial_2'"'] & C_1' \arrow[r, "\partial_1'"'] & C_0'
\end{tikzcd}$$

such that

$$f_0\partial_1 = \partial_1' f_1, \quad f_1\partial_2 = \partial_2' f_2$$

and such that

$$f_1(c_0 \cdot c_1) = f_0(c_0) \cdot f_1(c_1), \quad f_2(c_0 \cdot c_2) = f_0(c_0) \cdot f_2(c_2)$$

and

$$\{\otimes\} f_1 \otimes f_1 = f_2\{\otimes\}$$

for all $c_2 \in C_2$, $c_1 \in C_1$, $c_0 \in C_0$.

We denote the category of 2-crossed module by $\mathbf{X_2Mod}$.

In [4], the first author with Porter studied the truncated simplicial algebras and saw what properties that has. Later, they turned to a simplicial algebra $\mathbf{E}$ which is 2-truncated, i.e., its Moore complex look like;

$$\cdots \longrightarrow 0 \longrightarrow NE_2 \xrightarrow{\partial_2} NE_1 \xrightarrow{\partial_1} NE_0$$

and they then showed the following result:

Theorem 2.1 *The category* $\mathbf{X_2Mod}$ *of 2-crossed modules is equivalent to the category* $\mathbf{SimpAlg}_{\leqslant 2}$ *of simplicial algebras with Moore complex of length 2.*

3 2-Crossed Modules from Crossed Squares

Crossed squares were initially defined by Guin-Waléry and Loday in [18]. The commutative algebra analogue of crossed squares has been studied by Ellis (cf. [14]).

Now, we will recall the definition (due to Ellis [14]) of a crossed square of algebras. A *crossed square* is a commutative diagram

$$\begin{array}{ccc} L & \xrightarrow{\lambda} & M \\ {\scriptstyle\lambda'}\downarrow & & \downarrow{\scriptstyle\mu} \\ N & \xrightarrow[\nu]{} & R \end{array}$$

together with the actions of R on L, M and N. There are thus commutative actions of M on L and N via μ and a function

$$h : M \times N \to L$$

the h-map. This data must satisfy the following axioms:

1. The $\lambda, \lambda', \mu, \nu$ and $\mu\lambda = \nu\lambda'$ are crossed modules;
2. The maps λ, λ' preserve the action of R;
3. $kh(m, n) = h(km, n) = h(m, kn)$;
4. $h(m + m', n) = h(m, n) + h(m', n)$;
5. $h(m, n + n') = h(m, n) + h(m, n')$;
6. $r \cdot h(m, n) = h(r \cdot m, n) = h(m, r \cdot n)$;
7. $\lambda h(m, n) = m \cdot n$;
8. $\lambda' h(m, n) = n \cdot m$;
9. $h(m, \lambda' l) = m \cdot l$;
10. $h(\lambda l, n) = n \cdot l$

for all $l \in L, m, m' \in M$ and $n, n' \in N$, $r \in R$, $k \in \mathbf{k}$

We denote such a crossed square by (L, M, N, R). A morphism of crossed squares $\Phi : (L, M, N, R) \to (L', M', N', R')$ consists of homomorphisms

$$\begin{array}{cccccccc} \Phi_L : & L & \longrightarrow & L' & \quad \Phi_M : & M & \longrightarrow & M' \\ \Phi_N : & N & \longrightarrow & N' & \quad \Phi_R : & R & \longrightarrow & R' \end{array}$$

such that the cube of homomorphisms is commutative

$$\Phi_L h(m, n) = h(\Phi_M m, \Phi_N n)$$

with $m \in M$ and $n \in N$, and the homomorphisms Φ_L, Φ_M, Φ_N are Φ_R-equivariant. The category of crossed squares will be denoted by $\mathbf{Crs}^2$.

Conduché (in a private communication with Brown and (see also published version [13])) gives a construction of a 2-crossed module from a crossed square of groups. On the other hand, the first author gave a neat description of the passage from a crossed square to a 2-crossed module of algebras in [3]. He constructed a 2-crossed module from a crossed square of commutative algebras

$$\begin{array}{ccc} L & \xrightarrow{\lambda} & M \\ {\scriptstyle \lambda'}\downarrow & & \downarrow{\scriptstyle \mu} \\ N & \xrightarrow[\nu]{} & R \end{array}$$

as

$$L \xrightarrow{(-\lambda,\lambda')} M \rtimes N \xrightarrow{\mu+\nu} R$$

analogue to that given by Conduché in the group case (cf. [13]). This construction can be briefly summarised as follows:

Apply the nerve in the both directions so as to get a bisimplicial algebra, then apply either the diagonal or the Artin-Mazur codiagonal functor (cf. [2]) to get to a simplicial algebra and take the Moore complex. Consequently, he showed in [3, Proposition 5.1] that the Moore complex of this simplicial algebra is isomorphic to the mapping cone complex, in the sense of Loday, of the crossed square and this mapping cone has a 2-crossed module structure of algebras.

Note that the construction given by Arvasi summarised above preserves the homotopy modules. In fact, Ellis in [15] proved that the homotopies of the crossed square

$$\begin{array}{ccc} L & \xrightarrow{\lambda} & M \\ {\scriptstyle \lambda'}\downarrow & & \downarrow{\scriptstyle \mu} \\ N & \xrightarrow[\nu]{} & R \end{array}$$

are the homologies of the complex

$$L \xrightarrow{(-\lambda,\lambda')} M \rtimes N \xrightarrow{\mu+\nu} R \longrightarrow 0.$$

4 Crossed Squares from Simplicial Algebras

In 1991, Porter, [21], described a functor from the category of simplicial groups to that of crossed n-cubes defined by Ellis and Steiner in [16], based on ideas of Loday (cf. [20]). Crossed n-cubes in algebraic settings such as commutative algebras, Jordan algebras, Lie algebras have been defined by Ellis in [14]. The notion of crossed n-cube of commutative algebras defined by Ellis is undoubtedly important, however in this paper, we use only the case $n = 2$, that is for crossed squares. Hence, we do not remind the general definition of a crossed n-cube. From [14] we can say that, in low dimensions, a crossed 1-cube is the same as a crossed module and a crossed 2-cube is the same as a crossed square.

In [3], Arvasi adapt that description to give an obvious analogue of the functor given by Porter, [21], for the commutative algebra case.

In fact, the following result is the 2-dimensional case of a general construction of a crossed n-cube of algebras from a simplicial algebra given by Arvasi, [3], analogue to that given by Porter, [21], in the group case.

Let $\mathbf{E}$ be a simplicial algebra. Then the following diagram

$$\begin{tikzcd} NE_2/\partial_3 NE_3 \arrow[r, "\partial_2"] \arrow[d, "\partial_2'"'] & NE_1 \arrow[d, "\mu"] \\ \overline{NE}_1 \arrow[r, "\mu'"'] & E_1 \end{tikzcd}$$

is the underlying square of a crossed square. The extra structure is given as follows;
$NE_1 = \ker d_0^1$ and $\overline{NE}_1 = \ker d_1^1$. Since E_1 acts on $NE_2/\partial_3 NE_3, \overline{NE}_1$ and NE_1, there are actions of $\overline{NE}_1$ on $NE_2/\partial_3 NE_3$ and NE_1 via μ', and NE_1 acts on $NE_2/\partial_3 NE_3$ and $\overline{NE}_1$ via μ. Both μ and μ' are inclusions, and all actions are given by multiplication. The h-map is

$$\begin{array}{rccl} h: & NE_1 \times \overline{NE}_1 & \longrightarrow & NE_2/\partial_3 NE_3 \\ & (x, \overline{y}) & \longmapsto & h(x, \overline{y}) = s_1 x(s_1 y - s_0 y) + \partial_3 NE_3. \end{array}$$

Here x and y are in NE_1 as there is a natural bijection between NE_1 and $\overline{NE}_1$. This is clearly functorial and we denote it by;

$$\mathbf{M}(\,-\,,2) : \mathbf{SimpAlg} \longrightarrow \mathbf{Crs}^2.$$

5 Quadratic Modules from 2-Crossed Modules

As we mentioned in introduction, Baues defined the quadratic module of groups as an algebraic model of connected 3-types. In this section, we define the commutative algebra version of this structure and we define a functor from the category of 2-crossed modules to that of quadratic modules of algebras. We should recall some basic information before giving the definition of a quadratic module.

Recall that a *pre-crossed module* is a homomorphism $\partial : C \to R$ together with an action of R on C, written $c \cdot r$ for $r \in R$ and $c \in C$, satisfying the condition $\partial(c \cdot r) = \partial(c)r$ for all $r \in R$ and $c \in C$.

For an algebra C, C/C^2 is the quotient of the algebra C by its ideal of squares. Then, there is a functor from the category of $\mathbf{k}$-algebras to the category of $\mathbf{k}$-modules. This functor goes from C to C/C^2, plays the role of abelianization in the category of $\mathbf{k}$-algebras. As modules are often called singular algebras (e.g., in the theory of singular extensions) we shall call this functor "singularisation".

Now, we generalise these notions to the pre-crossed modules. Let $\partial : C \to R$ be a pre-crossed module and let $P_1(\partial) = C$ and let $P_2(\partial)$ be the Peiffer ideal of C generated by elements of the form

$$\langle x, y\rangle = xy - x \cdot \partial y$$

which is called the *Peiffer element* for $x, y \in C$.

A *nil(2)-module* is a pre-crossed module $\partial : C \to R$ with an additional "nilpotency" condition. This condition is $P_3(\partial) = 0$, where $P_3(\partial)$ is the ideal of C generated by Peiffer elements $\langle x_1, x_2, x_3\rangle$ of length 3.

The homomorphism

$$\partial^{cr} : C^{cr} = C/P_2(\partial) \longrightarrow R$$

is the crossed module associated to the pre-crossed module $\partial : C \to R$, since

$$\begin{aligned}
\langle [x], [y]\rangle &= (x + P_2(\partial))(y + P_2(\partial)) - (x + P_2(\partial)) \cdot \partial^{cr}(y + P_2(\partial)) \\
&= (xy + P_2(\partial)) - (x + P_2(\partial)) \cdot \partial^{cr}(y + P_2(\partial)) \\
&= xy - x \cdot \partial(y) + P_2(\partial) \\
&= P_2(\partial) \quad (\because \ \langle x, y\rangle \in P_2(\partial) \ ) \\
&= [0]
\end{aligned}$$

for $[x] = x + P_2(\partial)$, $[y] = y + P_2(\partial) \in C^{cr}$.

Now, we can give the definition of a quadratic module of algebras.

Definition 5.1 *A quadratic module $(\omega, \delta, \partial)$ is a diagram*

$$\begin{tikzcd}
 & C \otimes C \arrow[dl, "\omega"'] \arrow[d, "w"] & \\
L \arrow[r, "\delta"'] & M \arrow[r, "\partial"'] & N
\end{tikzcd}$$

of homomorphisms of algebras such that the following axioms are satisfied.

QM1)- *The homomorphism $\partial : M \to N$ is a nil(2)-module and the quotient map $M \twoheadrightarrow C = M^{cr}/(M^{cr})^2$ is given by $x \mapsto [x]$, where $[x] \in C$ denotes the class represented by $x \in M$. The map w is defined by Peiffer multiplication, i.e., $w([x] \otimes [y]) = xy - x \cdot \partial(y)$.*

QM2)- *The homomorphisms δ and ∂ satisfy $\delta\partial = 0$ and the quadratic map ω is a lift of the map w, that is $\delta\omega = w$ or equivalently*

$$\delta\omega([x] \otimes [y]) = w([x] \otimes [y]) = xy - x \cdot \partial(y)$$

for $x, y \in M$.

QM3)- *L is a N-algebra and all homomorphisms of the diagram are equivariant with respect to the action of N. Moreover, the action of N on L satisfies the following equality*

$$a \cdot \partial(x) = \omega([\delta a] \otimes [x] + [x] \otimes [\delta a])$$

for $a \in L, x \in N$.

QM4)- *For* $a, b \in L$,

$$\omega([\delta a] \otimes [\delta b]) = ab.$$

A map $\varphi : (\omega, \delta, \partial) \to (\omega', \delta', \partial')$ between quadratic modules is given by a commutative diagram, $\varphi = (l, m, n)$

$$\begin{tikzcd}
C \otimes C \arrow[r, "\omega"] \arrow[d, "\varphi_* \otimes \varphi_*"'] & L \arrow[r, "\delta"] \arrow[d, "l"'] & M \arrow[r, "\partial"] \arrow[d, "m"'] & N \arrow[d, "n"'] \\
C' \otimes C' \arrow[r, "\omega'"'] & L' \arrow[r, "\delta'"'] & M' \arrow[r, "\partial'"'] & N'
\end{tikzcd}$$

where (m, n) is a map between pre-crossed modules which induces $\varphi_* : C \to C'$ and where l is an n-equivariant homomorphism. Let **QM** be the category of quadratic modules and of maps as in above diagram.

Now, we construct a functor from the category of 2-crossed modules to the category of quadratic modules.

Let

$$C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

be a 2-crossed module of algebras. Let P_3 be the ideal of C_1 generated by elements of the form

$$\langle \langle x, y \rangle, z \rangle = \langle x, y \rangle z - \langle x, y \rangle \cdot \partial_1 z$$

and

$$\langle x, \langle y, z \rangle \rangle = x \langle y, z \rangle - x \cdot \partial_1(\langle y, z \rangle)$$

for $x, y, z \in C_1$. Let

$$q_1 : C_1 \to C_1/P_3$$

be a quotient map and let $M = C_1/P_3$ be the quotient algebra. Since ∂_1 is a pre-crossed module, we obtain

$$\begin{aligned}
\partial_1(\langle \langle x, y \rangle, z \rangle) &= \partial_1(\langle x, y \rangle z - \langle x, y \rangle \cdot \partial_1 z) \\
&= \partial_1((xy - x \cdot \partial_1 y)z - (xy - x \cdot \partial_1 y) \cdot \partial_1 z) \\
&= \partial_1 x \partial_1 y \partial_1 z - \partial_1 x \partial_1 y \partial_1 z - \partial_1 x \partial_1 y \partial_1 z + \partial_1 x \partial_1 y \partial_1 z \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\partial_1(\langle x, \langle y, z \rangle \rangle) &= \partial_1(x \langle y, z \rangle - x \cdot \partial_1 \langle y, z \rangle) \\
&= \partial_1(x(yz - y \cdot \partial_1 z) - x \cdot \partial_1(yz - y \cdot \partial_1 z)) \\
&= \partial_1(x(yz - y \cdot \partial_1 z) - x \cdot (\partial_1 y \partial_1 z - \partial_1 y \partial_1 z)) \\
&= \partial_1(x(yz - y \cdot \partial_1 z)) \\
&= \partial_1 x \partial_1 y \partial_1 z - \partial_1 x \partial_1 y \partial_1 z \\
&= 0
\end{aligned}$$

for $x, y, z \in C_1$. That is, $\partial_1(P_3) = 0$. Thus, the map $\partial : M \to C_0$ given by $\partial(x + P_3) = \partial_1(x)$, for all $x \in C_1$, is a well defined homomorphism since $\partial_1(P_3) = 0$. Therefore, we can write a commutative diagram

$$\begin{tikzcd} C_1 \arrow[rr, "\partial_1"] \arrow[dr, "q_1"'] & & C_0 \\ & M \arrow[ur, "\partial"'] & \end{tikzcd}$$

where $q_1 : C_1 \to M$ is the quotient map.

Let P_3' be the ideal of C_2 generated by elements of the form

$$\{x \otimes \langle y, z\rangle\} \text{ and } \{\langle x, y\rangle \otimes z\},$$

where $\{- \otimes -\}$ is the Peiffer lifting map. We have

$$L = C_2/P_3'.$$

We can write from **2CM1**)

$$\partial_2\{x \otimes \langle y, z\rangle\} = \langle x, \langle y, z\rangle\rangle$$

and

$$\partial_2\{\langle x, y\rangle \otimes z\} = \langle\langle x, y\rangle, z\rangle$$

and we thus obtain $\partial_2(P_3') = P_3$. Then, $\delta : L \longrightarrow M$ given by $\delta(l + P_3') = \partial_2 l + P_3$ is a well defined homomorphism. Indeed, if $l + P_3' = l' + P_3'$ then $l - l' \in P_3'$, and $\partial_2(l - l') \in \partial_2(P_3')$. Since $\partial_2(P_3') = P_3$, we obtain $\partial_2(l) - \partial_2(l') \in P_3$, that is, $\partial_2 l + P_3 = \partial_2 l' + P_3$.

Let

$$C = \frac{M^{cr}}{(M^{cr})^2}.$$

Thus we get the following commutative diagram;

$$\begin{tikzcd} & C \otimes C \arrow[dl, "\omega"'] \arrow[d, "w"] & \\ L \arrow[r, "\delta"] & M \arrow[r, "\partial"] & N \arrow[d, equal] \\ C_2 \arrow[u, "q_2"] \arrow[r, "\partial_2"'] & C_1 \arrow[u, "q_1"] \arrow[r, "\partial_1"'] & C_0 \end{tikzcd}$$

where q_1 and q_2 are the quotient maps. The quadratic map

$$\omega : C \otimes C \longrightarrow L$$

is given by the Peiffer lifting map, namely

$$\omega([q_1 x] \otimes [q_1 y]) = q_2\{x \otimes y\}$$

for all $x, y \in C_1$, $q_1 x, q_1 y \in M$ and $[q_1 x] \otimes [q_1 y] \in C \otimes C$.

Proposition 5.2 *The diagram*

$$\begin{tikzcd} & C \otimes C \arrow[dl, "\omega"'] \arrow[d, "w"] & \\ L \arrow[r, "\delta"'] & M \arrow[r, "\partial"'] & N \end{tikzcd}$$

is a quadratic module of algebras.

Proof: We show that all axioms of quadratic module are verified.

QM1)- Since the triple Peiffer elements in $M = C_1/P_3$ are trivial, the $\partial : M \to N$ is a $nil(2)$-module. Indeed, for $x + P_3$, $y + P_3$, $z + P_3 \in M$,

$$\begin{aligned} \langle x + P_3, \langle y + P_3, z + P_3 \rangle \rangle &= \langle x, \langle y, z \rangle \rangle + P_3 \\ &= 0 + P_3 \quad (\text{by } \langle x, \langle y, z \rangle \rangle \in P_3) \\ &= P_3 \end{aligned}$$

and

$$\begin{aligned} \langle \langle x + P_3, y + P_3 \rangle, z + P_3 \rangle &= \langle \langle x, y \rangle, z \rangle + P_3 \\ &= 0 + P_3 \quad (\text{by } \langle \langle x, y \rangle, z \rangle \in P_3) \\ &= P_3. \end{aligned}$$

QM2)- For $q_1x, q_1y \in M$ and $[q_1x] \otimes [q_1y] \in C \otimes C$, we have

$$\begin{aligned} \delta\omega \left([q_1x] \otimes [q_1y] \right) &= \delta q_2\{x \otimes y\} \\ &= q_1\partial_2\{x \otimes y\} \\ &= q_1xq_1y - q_1x \cdot \partial(q_1y) \quad (\text{by } \mathbf{2CM1})) \\ &= w([q_1x] \otimes [q_1y]). \end{aligned}$$

QM3)- For $q_2a \in L$ and $q_1x \in M$, and $[\delta q_2a] \otimes [q_1x] \in C \otimes C$, we have

$$\begin{aligned} \omega \left([\delta q_2a] \otimes [q_1x] \right) &= \omega \left([q_1\partial_2a] \otimes [q_1x] \right) \\ &= q_2\{\partial_2a \otimes x\} \\ &= -q_2a \cdot x + q_2a \cdot \partial q_1(x) \quad (\text{by } \mathbf{2CM4})) \end{aligned}$$

and

$$\begin{aligned} \omega \left([q_1x] \otimes [\delta q_2a] \right) &= \omega \left([q_1x] \otimes [q_1\partial_2a] \right) \\ &= q_2\{x \otimes \partial_2a\} \\ &= q_2a \cdot x \quad (\text{by } \mathbf{2CM4})). \end{aligned}$$

Then, we have

$$\omega \left([\delta q_2a] \otimes [q_1x] + [q_1x] \otimes [\delta q_2a] \right) = q_2a \cdot \partial q_1(x).$$

QM4)- For $q_2a, q_2b \in L$, and $[\delta q_2a] \otimes [\delta q_2b] \in C \otimes C$, we have

$$\begin{aligned} \omega \left([\delta q_2a] \otimes [\delta q_2b] \right) &= \omega \left([q_1\partial_2a] \otimes [q_1\partial_2b] \right) \\ &= q_2\{\partial_2a \otimes \partial_2b\} \\ &= (q_2a)(q_2b) \quad (\text{by } \mathbf{2CM2})). \end{aligned}$$

$\square$

Therefore, we have defined a functor from the category of 2-crossed modules to that of quadratic modules of algebras. We denote it by

$$\Lambda : \mathbf{X_2Mod} \longrightarrow \mathbf{QM}.$$

Now, we show that the functor Λ described above preserves the homotopy modules.

Proposition 5.3 *Let*

$$C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

be the 2-crossed module and π_i be its homotopy modules for all $i \geqslant 0$. Let π_i' be the homotopy modules of its associated quadratic module

$$\begin{tikzcd} & C \otimes C \arrow[dl, "\omega"'] \arrow[d, "w"] & \\ L \arrow[r, "\delta"'] & M \arrow[r, "\partial"'] & N. \end{tikzcd}$$

Then, $\pi_i \cong \pi_i'$ for all $i \geqslant 0$.

Proof: The homotopy modules of the 2-crossed module are

$$\pi_i = \begin{cases} C_0/\partial_1(C_1) & i = 1, \\ \ker \partial_1/\mathrm{Im}\partial_2 & i = 2, \\ \ker \partial_2 & i = 3, \\ 0 & i = 0,\ i > 3 \end{cases}$$

and the homotopy modules of its associated quadratic module are

$$\pi_i' = \begin{cases} N/\partial(M) & i = 1, \\ \ker \partial/\mathrm{Im}\delta & i = 2, \\ \ker \delta & i = 3, \\ 0 & i = 0,\ i > 3. \end{cases}$$

Now, we show that $\pi_i \cong \pi_i'$ for all $i \geqslant 0$. Since $\partial_1(P_3) = 0$, $M = C_1/P_3$, $N = C_0$ and $\partial(M) \cong \partial_1(C_1)$, clearly we have

$$\pi_1 = C_0/\partial_1(C_1) \cong N/\partial(M) = \pi_1'.$$

Since $\ker \partial \cong \dfrac{\ker \partial_1}{P_3}$ and $\mathrm{Im}\delta \cong \dfrac{\mathrm{Im}\partial_2}{P_3}$ so that we have

$$\pi_2' = \frac{\ker \partial}{\mathrm{Im}\delta} \cong \frac{\ker \partial_1/P_3}{\mathrm{Im}\partial_2/P_3} \cong \frac{\ker \partial_1}{\mathrm{Im}\partial_2} = \pi_2.$$

Now, we show that $\pi_3 \cong \pi'_3$. Consider that

$$\pi'_3 = \{x + P'_3 : \partial_2(x) \in P_3\}.$$

For an element $x + P'_3$ of π'_3, we show that there is an element $x' + P'_3$ of π'_3 such that $x + P'_3 = x' + P'_3$ and $x' \in \ker \partial_2$. In fact, we observe from **2CM1**) that $\partial_2\{\langle x, y\rangle \otimes z\} = \langle \langle x, y\rangle, z\rangle$ and $\partial_2\{x \otimes \langle y, z\rangle\} = \langle x, \langle y, z\rangle\rangle$, and we have $\partial_2(P'_3) = P_3$. Hence $\partial_2(x) \in P_3$ implies $\partial_2(x) = \partial_2(w)$, $w \in P'_3$; thus $\partial_2(x - w) = 0$; then take $x' = x - w$, so that $x + P'_3 = x' + P'_3$ and $\partial_2(x') = 0$. Define $\mu : \pi'_3 \to \pi_3$ by $(x' + P'_3) \mapsto x'$ and $\nu : \pi_3 \to \pi'_3$ by $x \mapsto x + P_3$. Then μ and ν are inverse bijections, that is, we have $\pi_3 \cong \pi'_3$. Thus, the homotopy modules of the 2-crossed module are isomorphic to that of its associated quadratic module. $\square$

6 Quadratic Modules from Simplicial Algebras

Baues, [6], defined a functor from simplicial groups to quadratic modules. In the construction of this functor, to define the quadratic map ω, Baues in [6] used the Conduché's Peiffer lifting map $\{-, -\}$ given for the construction of a 2-crossed module from a simplicial group (cf. [12]).

In this section, we construct a functor from the category of simplicial algebras to the category of quadratic modules in terms of hypercrossed complex pairings analogously to that given by Baues in [6]. In the construction, to define the quadratic map ω, we use the Peiffer lifting map given by Arvasi and Porter in [4, Proposition 5.2].

Now, we will construct a functor from simplicial algebras to quadratic modules by using the $C_{\alpha,\beta}$ functions. We will use the $C_{\alpha,\beta}$ functions in verifying the axioms of quadratic module.

Let $\mathbf{E}$ be a simplicial algebra with Moore complex $\mathbf{NE}$. We will obtain a quadratic module by using the following commutative diagram

$$\begin{tikzcd} & NE_1 \times NE_1 \arrow[dl, "\omega'"'] \arrow[d, "w"] & \\ NE_2/\partial_3(NE_3) \arrow[r, "\overline{\partial_2}"'] & NE_1 \arrow[r, "\partial_1"'] & NE_0 \end{tikzcd}$$

where the map w is given by

$$w(x, y) = xy - x \cdot \partial_1 y$$

for $x, y \in NE_1$ and the map ω' is given by

$$\omega'(x, y) = \overline{s_1x(s_1y - s_0y)} = s_1x(s_1y - s_0y) + \partial_3(NE_3)$$

for $x, y \in NE_1$.

Let $P_3(\partial_1)$ be the ideal of NE_1 generated by elements of the form

$$\langle x, \langle y, z\rangle\rangle \text{ and } \langle \langle x, y\rangle, z\rangle$$

for $x, y, z \in NE_1$. Since ∂_1 is a pre-crossed module, we can write $\partial_1(P_3(\partial_1)) = 0$. Then, $\partial : NE_1/P_3(\partial_1) \longrightarrow NE_0$ given by $\partial(x + P_3(\partial_1)) = \partial_1 x$ for $x \in NE_1$ is a well defined homomorphism. Thus, we obtain the following commutative diagram

$$\begin{tikzcd} NE_1 \arrow[rr, "\partial_1"] \arrow[dr, "q_1"'] & & NE_0 \\ & NE_1/P_3(\partial_1) \arrow[ur, "\partial"'] & \end{tikzcd}$$

where q_1 is the quotient map.

Let $P_3'(\partial_1)$ be an ideal of $NE_2/\partial_3(NE_3)$ generated by the formal Peiffer elements of x, y, z, i.e., generated by elements of the form

$$\omega'(\langle x, y\rangle, z) = s_1(\langle x, y\rangle)(s_1 z - s_0 z)$$

and

$$\omega'(x, \langle y, z\rangle) = s_1(x)(s_1(\langle y, z\rangle) - s_0(\langle y, z\rangle))$$

for $x, y, z \in NE_1$. Notice that

$$\begin{aligned} \overline{\partial_2}\omega'(x, \langle y, z\rangle) &= d_2(s_1(x)(s_1(\langle y, z\rangle) - s_0(\langle y, z\rangle))) \\ &= x(\langle y, z\rangle) - x s_0 d_1(\langle y, z\rangle) \\ &= \langle x, \langle y, z\rangle\rangle \end{aligned}$$

and

$$\begin{aligned} \overline{\partial_2}\omega'(\langle x, y\rangle, z) &= d_2(s_1(\langle x, y\rangle)(s_1(z) - s_0(z))) \\ &= \langle x, y\rangle (z - s_0 d_1 z) \\ &= \langle\langle x, y\rangle, z\rangle . \end{aligned}$$

We obtain $\overline{\partial_2}(P_3'(\partial_1)) = P_3(\partial_1)$. Let

$$M = NE_1/P_3(\partial_1)$$

and

$$L = (NE_2/\partial_3(NE_3))/P_3'(\partial_1)$$

be the quotient algebras. We thus see that the map

$$\delta : (NE_2/\partial_3 NE_3))/P_3'(\partial_1) \longrightarrow NE_1/P_3(\partial_1)$$

given by $\delta(a + P_3'(\partial_1)) = \overline{\partial_2}(a) + P_3(\partial_1)$ is a well defined homomorphism since $\overline{\partial_2}(P_3'(\partial_1)) = P_3(\partial_1)$.

We obtain the following commutative diagram,

$$\begin{tikzcd} & C \otimes C \arrow[dl, "\omega"'] \arrow[d, "w"] & \\ L \arrow[r, "\delta"'] & M \arrow[r, "\partial"'] & N \arrow[d, equal] \\ NE_2/\partial_3(NE_3) \arrow[u, "q_2"] \arrow[r, "\overline{\partial_2}"'] & NE_1 \arrow[u, "q_1"'] \arrow[r, "\partial_1"'] & NE_0 \end{tikzcd}$$

where $C = M^{cr}/(M^{cr})^2$ and q_1, q_2 are the quotient maps and the quadratic map ω can be given by

$$\omega([q_1x] \otimes [q_1y]) = q_2\omega'(x, y) = q_2\left(s_1x(s_1y - s_0y) + \partial_3(NE_3)\right)$$

for $q_1x, q_1y \in M$ and $[q_1x] \otimes [q_1y] \in C \otimes C$.

Proposition 6.1 *The diagram*

$$\begin{tikzcd}
 & C \otimes C \arrow[dl, "\omega"'] \arrow[d, "w"] & \\
L \arrow[r, "\delta"'] & M \arrow[r, "\partial"'] & N
\end{tikzcd}$$

is a quadratic module of algebras.

Proof: We show that all axioms of quadratic module are verified by using the $C_{\alpha,\beta}$ functions. We display the elements omitting the overlines in our calculation to save complication.

QM1)- Clearly, $\partial : M \to N$ is a nil(2)-module. Because, for $x + P_3(\partial_1)$, $y + P_3(\partial_1)$, $z + P_3(\partial_1) \in M = NE_1/P_3(\partial_1)$,

$$\begin{aligned}
\langle x + P_3(\partial_1), \langle y + P_3(\partial_1), z + P_3(\partial_1)\rangle\rangle &= \langle x, \langle y, z\rangle\rangle + P_3(\partial_1) \\
&= 0 + P_3(\partial_1) \text{ (by } \langle x, \langle y, z\rangle\rangle \in P_3(\partial_1)) \\
&= P_3(\partial_1)
\end{aligned}$$

and similarly

$$\begin{aligned}
\langle\langle x + P_3(\partial_1), y + P_3(\partial_1)\rangle, z + P_3(\partial_1)\rangle &= \langle\langle x, y\rangle, z\rangle + P_3(\partial_1) \\
&= 0 + P_3(\partial_1) \ (\because \ \langle x, \langle y, z\rangle\rangle \in P_3(\partial_1)) \\
&= P_3(\partial_1).
\end{aligned}$$

QM2)- For $x, y \in NE_1$, $q_1x, q_1y \in M$ and $[q_1x] \otimes [q_1y] \in C \otimes C$, we have

$$\begin{aligned}
\delta\omega([q_1x] \otimes [q_1y]) &= \delta q_2\left(s_1x(s_1y - s_0y)\right) \\
&= q_1(d_2(s_1x(s_1y - s_0y))) \quad (\because \ \delta q_2 = q_1\overline{\partial_2}) \\
&= q_1(x)q_1(y) - q_1(x) \cdot \overline{\partial_1}(q_1y) \quad (\because \ \partial q_1 = d_1) \\
&= w([q_1x] \otimes [q_1y]).
\end{aligned}$$

QM3)- For $q_2a \in L$ and $q_1x \in M$, we have

$$\omega\left([\delta q_2a] \otimes [q_1x]\right) = \omega\left([q_1\overline{\partial_2}a] \otimes [q_1x]\right)$$

and

$$\omega\left([q_1\overline{\partial_2}a] \otimes [q_1x]\right) = q_2(s_1d_2a(s_1x - s_0x)).$$

On the other hand, from

$$\partial_3(C_{(2,0)(1)}(x \otimes a)) = (s_0x - s_1x)s_1d_2a - (s_0x - s_1x)a \in \partial_3(NE_3)$$

we have

$$\begin{aligned}\omega\left([q_1\overline{\partial_2}a]\otimes[q_1x]\right) &\equiv s_0d_1x(q_2a)-s_1x(q_2a) \quad \bmod(\partial_3(NE_3))\\ &= (q_2a)\cdot\overline{\partial_1}q_1x - x\cdot(q_2a) \quad (\because\ \overline{\partial_1}q_1=d_1).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\omega\left([q_1x]\otimes[\delta q_2a]\right) &= \omega\left([q_1x]\otimes[q_1\overline{\partial_2}a]\right)\\ &= q_2(s_1x(s_1d_2a-s_0d_2a)).\end{aligned}$$

From

$$\partial_3(C_{(2,1)(0)}(x\otimes a)) = s_1x(s_0d_2a-s_1d_2a)+s_1xa\in\partial_3(NE_3)$$

we can write

$$\omega\left([q_1x]\otimes[q_1\overline{\partial_2}a]\right) \equiv x\cdot(q_2a) \quad \bmod(\partial_3(NE_3)).$$

Therefore,

$$\omega([\delta q_2a]\otimes[q_1x]+[q_1x]\otimes[\delta q_2a]) = (q_2a)\cdot\partial(q_1x).$$

QM4)- For $q_2a, q_2b\in L$, we can write

$$\omega\left([\delta q_2a]\otimes[\delta q_2b]\right) = q_2(s_1d_2a(s_1d_2b-s_0d_2b)).$$

From

$$\partial_3(C_{(1)(0)}(a\otimes b)) = s_1d_2a(s_0d_2b-s_1d_2b)+ab\in\partial_3(NE_3)$$

we obtain

$$\omega\left([q_1\overline{\partial_2}a]\otimes[q_1\overline{\partial_2}b]\right) \equiv q_2(a)q_2(b) \quad \bmod(\partial_3(NE_3)).$$

$\square$

Therefore, we defined a functor from the category of simplicial algebras to that of quadratic modules of algebras. We denote it by

$$\Delta:\mathbf{SimpAlg}\longrightarrow\mathbf{QM}.$$

Alternatively, this proposition can be reproved differently, by making use of the 2-crossed module constructed from a simplicial algebra by Arvasi and Porter (cf. [4]). We now give a sketch of the argument. In [4, Proposition 5.2], it is shown that given a simplicial algebra $\mathbf{E}$, one can construct a 2-crossed module

$$NE_2/\partial_3(NE_3\cap D_3)\xrightarrow{\overline{\partial_2}}NE_1\xrightarrow{\partial_1}NE_0 \tag{1}$$

where $\{x\otimes y\} = s_1x(s_1y-s_0y)+\partial_3(NE_3\cap D_3)$ for $x,y\in NE_1$.
Clearly we have a commutative diagram

$$\begin{array}{ccccc} NE_2/\partial_3(NE_3\cap D_3) & \xrightarrow{\overline{\partial_2}} & NE_1 & \xrightarrow{\partial_1} & NE_0\\ \Big\downarrow{\scriptstyle j} & & \| & & \|\\ NE_2/\partial_3(NE_3) & \longrightarrow & NE_1 & \xrightarrow{\partial_1} & NE_0. \end{array}$$

Consider now the quadratic module associated to the 2-crossed module (1), as in Section 5 of this paper.

$$\begin{tikzcd} & & C\otimes C \arrow[dll, "\omega'"'] \arrow[d] & \\ L' \arrow[rr, "\delta'"] & & M \arrow[r] & N \arrow[d, equal] \\ \frac{NE_2}{\partial_3(NE_3\cap D_3)} \arrow[u, "q_2"] \arrow[rr, "\overline{\partial_2}"'] & & NE_1 \arrow[u] \arrow[r, "\partial_1"'] & NE_0 \end{tikzcd}$$

Then one can see that $L' = A/\partial_3(NE_3\cap D_3)$, where A is the ideal of NE_2 generated by elements of the form

$$s_1(\langle x,y\rangle)(s_1z - s_0z) \text{ and } s_1(x)(s_1(\langle y,z\rangle) - s_0(\langle y,z\rangle)).$$

On the other hand we have, from Section 5, $L = A/\partial_3(NE_3)$. Hence there is a map $i : L' \to L$ with

$$\omega = i\omega', \quad \delta' = \delta i. \tag{2}$$

Since

$$C\otimes C \xrightarrow{\omega'} L' \xrightarrow{\delta'} M \xrightarrow{\partial} N$$

is, by construction a quadratic module, it is straightforward to check, using (2), that

$$C\otimes C \xrightarrow{\omega} L \xrightarrow{\delta} M \xrightarrow{\partial} N$$

is also a quadratic module.

Now we show that the functor Δ described above preserves the homotopy modules.

Proposition 6.2 *Let $\mathbf{E}$ be a simplicial algebra, let π_i be the homotopy modules of the classifying space of $\mathbf{E}$ and let π_i' be the homotopy modules of its associated quadratic module; then $\pi_i \cong \pi_i'$ for $i = 0,1,2,3$.*

Proof: Let $\mathbf{E}$ be a simplicial algebra. The nth homotopy module of $\mathbf{E}$ is isomorphic to the nth homology of the Moore complex of $\mathbf{E}$, i.e., $\pi_n(\mathbf{E}) \cong H_n(\mathbf{NE})$.Thus the homotopy modules $\pi_n(\mathbf{E}) = \pi_n$ of $\mathbf{E}$ are

$$\pi_n = \begin{cases} NE_0/d_1(NE_1) & n = 1, \\ \dfrac{\ker d_1 \cap NE_1}{d_2(NE_2)} & n = 2, \\ \dfrac{\ker d_2 \cap NE_2}{d_3(NE_3)} & n = 3, \\ 0 & n = 0, \text{or } n > 3 \end{cases}$$

and the homotopy modules π'_n of its associated quadratic module are

$$\pi'_n = \begin{cases} NE_0/\partial(M) & n = 1, \\ \ker\partial/\mathrm{Im}\delta & n = 2, \\ \ker\delta & n = 3, \\ 0 & n = 0, \text{or } n > 3. \end{cases}$$

We claim that $\pi'_n \cong \pi_n$ for $n = 1, 2, 3$. Since $M = NE_1/P_3(\partial_1)$ and $\partial_1(P_3(\partial_1)) = 0$, we have

$$\partial(M) = \partial(NE_1/P_3(\partial_1)) = d_1(NE_1)$$

and then

$$\pi'_1 = NE_0/\partial(M) \cong NE_0/d_1(NE_1) = \pi_1.$$

Also $\ker\partial = \dfrac{\ker d_1 \cap NE_1}{P_3(\partial_1)}$ and $\mathrm{Im}\delta = d_2(NE_2)/P_3(\partial_1)$ so that we have

$$\pi'_2 = \frac{\ker\partial}{\mathrm{Im}\,\delta} = \frac{(\ker d_1 \cap NE_1)/P_3(\partial_1)}{d_2(NE_2)/P_3(\partial_1)} \cong \frac{\ker d_1 \cap NE_1}{d_2(NE_2)} = \pi_2.$$

The isomorphism between π'_3 and π_3 can be proved similarly to the proof of Proposition 5.3.

$\square$

Remark: Note that in the previous section, we have defined the functor Λ from the category of 2-crossed modules to that of quadratic modules. As mentioned in Section 2 and above, Arvasi and Porter, [4], constructed an equivalence between simplicial algebras with Moore complex of length 2 and 2-crossed modules of algebras. We can summarise these statements in the following diagram

$$\begin{tikzcd} \mathbf{X_2Mod} \arrow[rr, leftrightarrow, "{[4]}"] \arrow[dr, "\Lambda"'] & & \mathbf{SimpAlg}_{\leqslant 2} \arrow[dl, "\Delta"] \\ & \mathbf{QM} & \end{tikzcd}$$

7 Quadratic Modules from Crossed Squares

In this section we will define a functor from crossed squares to quadratic modules of algebras. Our construction can be briefly explained as:

Given a crossed square of algebras, we consider the associated 2-crossed module from Section 3 (cf. [3]), and then we build the quadratic module corresponding to this 2-crossed module as in Section 5. In other words, we are just composing two functors. In particular, the homotopy type is clearly preserved, as it is preserved at each step.

Now, recall that the first author in [3] constructed a 2-crossed module from a crossed square of commutative algebras

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & R \end{tikzcd}$$

as

$$L \xrightarrow{(-\lambda,\lambda')} M \rtimes N \xrightarrow{\mu+\nu} R \tag{3}$$

analogue to that given by Conduché in the group case (cf. [13]), as we mention in Section 3.

Now let

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & R \end{tikzcd}$$

be a crossed square of algebras. Consider its associated 2-crossed module (3). From this 2-crossed module, we can construct a quadratic module as in Section 5,

$$\begin{tikzcd} & C \otimes C \arrow[dl, "\omega"'] \arrow[d, "w"] & \\ L' \arrow[r, "\delta"'] & M' \arrow[r, "\partial"'] & N \end{tikzcd}$$

where $N = R$, $M' = (M \rtimes N)/P_3$, $L' = L/P_3'$, $C = (M')^{cr}/((M')^{cr})^2$.

The Peiffer elements in $M \rtimes N$ are given by

$$\begin{aligned} \langle (m,n),(c,a)\rangle &= (m,n)(c,a) - (m,n)\cdot(\mu(c)+\nu(a)) \\ &= (mc + m\cdot\nu(a) + c\cdot\nu(n), na) \\ &\quad -(m\cdot\mu(c) + m\cdot\nu(a), n\cdot\mu(c) + n\cdot\nu(a)) \\ &= (c\cdot\nu(n), -n\cdot\mu(c)). \end{aligned}$$

We know from Section 5 that the ideal P_3 of $M \rtimes N$ is generated by elements of the form

$$\left\langle \langle (m,n),(c,a)\rangle, (m',n')\right\rangle = (-m'\cdot\nu(n\cdot\mu(c)), \mu(m')\cdot(n\cdot\mu(c)))$$

and

$$\left\langle (m,n), \left\langle (c,a),(m',n')\right\rangle\right\rangle = ((m'\cdot\nu(a))\cdot\nu(n), -n\cdot\mu(m'\cdot\nu(a)))$$

for $(m,n),(c,a),(m',n') \in M \rtimes N$. We thus have $(\mu+\nu)(P_3) = 0$ and the map $\partial : (M \rtimes N)/P_3 \to R$ is given by $\partial((m,n)+P_3) = \mu(m)+\nu(n)$.

We also know that the ideal P_3' of L is generated by elements of the form

$$h(m', -n\cdot\mu(c)) \text{ and } h(m'\cdot\nu(a), n)$$

for all $m', c \in M$ and $n, a \in N$. We have

$$\begin{aligned} (-\lambda,\lambda')(h(m', -n\cdot\mu(c))) &= (-m'\cdot\nu(n\cdot\mu(c)), \mu(m')\cdot(n\cdot\mu(c))) \\ &= \langle\langle (m,n),(c,a)\rangle, (m',n')\rangle \end{aligned}$$

and

$$\begin{aligned} (-\lambda,\lambda')(h(m'\cdot\nu(a), n)) &= ((m'\cdot\nu(a))\cdot\nu(n), -n\cdot\mu(m'\cdot\nu(a))) \\ &= \langle (m,n), \langle (c,a),(m',n')\rangle\rangle \end{aligned}$$

by the crossed square axioms. Thus the map $\delta : L' \to M'$ is given by $\delta\left(l + P_3'\right) = (-\lambda l, \lambda' l) + P_3$ for all $l \in L$. The quadratic map

$$\omega : C \otimes C \longrightarrow L'$$

can be given by

$$\omega\left([q_1(m,n)] \otimes [q_1(c,a)]\right) = q_2(h(c,na))$$

for all $(m,n),(c,a) \in M \rtimes N$, $q_1(m,n)$, $q_1(c,a) \in M'$ and $[q_1(m,n)] \otimes [q_1(c,a)] \in C \otimes C$, and where h is the h-map of the crossed square.

Proposition 7.1 *The diagram*

$$\begin{tikzcd} & C \otimes C \arrow[dl, "\omega"'] \arrow[d, "w"] & \\ L' \arrow[r, "\delta"'] & M' \arrow[r, "\partial"'] & N \end{tikzcd}$$

is a quadratic module of algebras.

Proof: We show that all axioms of quadratic module are verified.

QM1)- Obviously $\partial : M' \to N$ is a $nil(2)$-module, since the triple Peiffer elements in M' are trivial. It can be proved similarly to the proof of Proposition 5.2.

QM2)- For $q_1(m,n)$ and $q_1(c,a) \in M'$ and $[q_1(m,n)], [q_1(c,a)] \in C$

$$\begin{aligned} \delta\omega\left([q_1(m,n)] \otimes [q_1(c,a)]\right) &= (-\lambda, \lambda')q_2(h(c,na)) \\ &= q_1((-\lambda h(c,na), \lambda' h(c,na)) \\ &= w([q_1(m,n)] \otimes [q_1(c,a)]). \end{aligned}$$

QM3)- For $q_1(c,a), \delta(q_2 l) \in M'$ and $[\delta q_2 l], [q_1(c,a)] \in C$,

$$\begin{aligned} \omega([\delta q_2 l] \otimes [q_1(c,a)]) + \omega([q_1(c,a)] \otimes [\delta q_2 l]) &= \omega[q_1(-\lambda l, \lambda' l)] \otimes [q_1(c,a)] + \omega[q_1(c,a)] \otimes [q_1(-\lambda l, \lambda' l)] \\ &= q_2(h(c,(\lambda' l)a)) + q_2(-\lambda l, a\lambda' l) \\ &= q_2(h(c, \lambda'(l \cdot a)) + q_2(h(-\lambda l, \lambda'(l \cdot a))) \\ &= (\mu(c) + \nu(a)) \cdot q_2(l). \end{aligned}$$

QM4)- For $\delta(q_2 l) = q_1(-\lambda l, \lambda' l)$ and $\delta(q_2 l_0) = q_1(-\lambda l_0, \lambda' l_0) \in M'$ and $[\delta q_2 l], [\delta q_2 l_0] \in C$,

$$\begin{aligned} \omega\left([\delta q_2 l] \otimes [\delta q_2 l_0]\right) &= \omega[q_1(-\lambda l, \lambda' l)] \otimes [q_1(-\lambda l_0, \lambda' l_0)] \\ &= q_2(h(\lambda l, \lambda'(l l_0))) \\ &= (q_2 l)(q_2 l_0). \end{aligned}$$

$\square$

Thus, we would have defined a functor from the category of crossed squares to that of quadratic modules of commutative algebras and we denote it by

$$\Psi : \mathbf{Crs}^2 \longrightarrow \mathbf{QM}.$$

Thus, the diagram given in introduction was completed.

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Z. Arvasi
Eskişehir Osmangazi University
Science Faculty
Department of Mathematics
26480, Eskişehir-TURKEY
e-mail: zarvasi@ogu.edu.tr

E. Ulualan
Dumlupınar University
Science Faculty
Department of Mathematics
Kütahya-TURKEY
eulualan@dumlupinar.edu.tr